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Global spinors and orientable five-branes

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Abstract: Fermion fields on an M-theory five-brane carry a representation of the double cover of the structure group of the normal bundle. It is shown that, on an arbitrary oriented Lorentzian six-manifold, there is always an Sp_2 twist that allows such spinors to be defined globally. The vanishing of the arising potential obstructions does not depend on spin structure in the bulk, nor does the six-manifold need to be spin or spin$^{\mathbb{C}}$. Lifting the tangent bundle to such a generalised spin bundle requires picking a generalised spin structure in terms of certain elements in the integral and modulo-two cohomology of the five-brane world-volume in degrees four and five, respectively.

Keywords: Extended Supersymmetry, M-Theory, p-branes, Differential and Algebraic Geometry.

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Contents

1. Introduction

The purpose of this paper is to verify that there are no obstructions for defining global Fermion fields on an orientable Lorentzian six-manifold, when the Fermions carry Sp_2 charge. This Fermion structure arises in the chiral $N = 2$ tensor theory in six dimensions [1]. Though this is the world-volume theory on an M-theory five-brane [2−4], the analysis presented here does not make use of assumptions relating to how or even whether the theory occurs on a brane embedded in an ambient Supergravity space-time [5].

The spin group $\mathrm{Spin}(1,5)$ can be thought of as an extension of the frame group $\mathrm{SO}(1,5)$ by $\mathbb{Z}_2$. The fact that the Fermions in the present case carry Sp_2 charge, leads to extensions of the frame group $\mathrm{SO}(1,5)$ by Sp_2, rather than the usual $\mathbb{Z}_2$. Though such extensions can contain both Sp_2 and $\mathrm{Spin}(1,5)$ as subgroups, this is a rather more general construction [6].

The detailed investigation of the actual lifting procedure allows one to find which choices precisely need to be made in order to specify a generalised spin structure. It turns out that there is latitude on two different levels: different ways of finding a lift correspond to a pair of elements, one from $H^4(M,\mathbb{Z})$ and the other from $H^5(M,\mathbb{Z}_2)$, where M is the six-manifold under study. Such a pair of elements can be thought of as (a model of) generalised spin structure. An element chosen from the former group has an interpretation in terms of a characteristic class related to the normal bundle, whereas an element from the latter group has no immediate such meaning.

There are several reasons to study the five-brane world-volume theory on its own right as a chiral $N = 2$ tensor theory in six dimensions [1]. For instance, in a situation where the brane configuration is not a stable solitonic solution, the way the brane is embedded in a geometric bulk space (if applicable) could be subject to significant dynamical corrections, as the quantum effects on transverse scalars on a brane could correct semiclassical brane geometry. In a certain sense, branes have the capacity to reconstruct their own transverse space. This is particularly relevant in the study of unstable brane configurations.

The relationship between the five-brane world-volume theory to the Supergravity Theory in the bulk is subtle. Verifying that local anomalies cancel on the five-brane seems to require an anomaly inflow mechanism, where the precise way in which the five-brane is embedded in the ambient space-time plays a pivotal rôle [7]. However, the actual process that guarantees this cancellation involves cutting quite concretely the brane off the background. Nevertheless, in a certain large N limit, the world-volume theory seems to lead to an equivalence beyween the two theories, as the superconformal theory based on the chiral $N = 2$ tensor theory in six dimensions provides a Holographic dual to Supergravity on $AdS_7 \times S^4$, as conjectured in [8].

All of this leaves much space for clarification as to the precise relationship of the five-brane world-volume to the 11-dimensional background. Especially unstable backgrounds in mind, it is important to divorce world-volume dynamics from bulk phenomena, as the former could then be used to probe quantum mechanical backgrounds that are more general than the smooth orientable spin-manifolds that usually appear in Supergravity backgrounds.

The structure of generalised spin groups is of course interesting on its on right, and the present analysis should provide a useful case study.

The paper is organised as follows: In section 2 the form of the extension $\mathrm{Spin}_G(1,5)$ that the paper concentrates on is specified. This is the only piece of information from the bulk that is used, and is forced on one already by the local world-volume theory. In section 3 the integral and modulo-two cohomology of the classifying space of generalised spin structures is found. The general lifting procedure is outlined in terms of Postnikov-Moore systems in section 4; this structure is used in section 5 to show the absence of obstructions, and to find the cohomology groups where generalised spin structures are classified. Though the preceding analysis does not depend on the compactness of the underlying six-manifold, the vanishing of obstructions and the classification of spin structures in different non-compact cases is commented on in section 6. In the final section, section 7, interpretation of the data required for a generalised spin structure, consequences of the brane being smoothly embedded in a spin manifold, and open questions are considered.

2. Five-brane structure

In this section the topological constraints on a stable supersymmetric five-brane, embedded in an 11-dimensional background are reviewed. This sheds light on the local Fermion structure as well.

The 11-dimensional Supergravity theory [5] can be formulated on orientable Lorentzian manifolds X that have a spin structure [3]. Orientability is required for a Lagrangian formulation; spin structure is required for defining global gravitino fields. Topologically this means that the first two Stiefel-Whitney classes of the tangent bundle of that manifold must vanish

$$w_1(TX) = w_2(TX) = 0\,. \tag{2.1}$$

In this geometric sense a five-brane — as a stable, supersymmetric solitonic solution of Supergravity — can be thought of as a smooth Lorentzian submanifold $\iota : M \hookrightarrow X$ embedded in the 11-dimensional background space X. This means that the tangent and the normal bundles are related as

$$\iota^* TX = TM \oplus NM\,, \tag{2.2}$$

and the above restrictions (2.1) on the bulk space-time imply

$$w_1(NM) = w_1(TM) \tag{2.3}$$

$$w_2(NM) = w_2(TM) + w_1(TM)^2\,. \tag{2.4}$$

For a Lagrangian formulation, one assumes again that the five-brane is orientable

$$w_1(TM) = w_1(NM) = 0\,. \tag{2.5}$$

Defining a world-volume theory on a chiral five-brane is subtle [9], first of all as the theory involves a self-dual three-form field strength [1]. Apart from this tensor field, the world-volume supports also scalar fields and spinorial fields, both of which have to exists as global fields on the brane.

Spinorial structures enter the *local* theory on the five-brane in the following way [1]:

- The transverse scalars transform, by definition, as a fundamental $\underline{\mathbf{5}}$ of the transverse SO(5). In the supersymmetric theory on the brane, however, they show up in the antisymmetric $\underline{\mathbf{5}}$ of Sp_2.

- The chiral Spin$(1,5)$-spinors belong to the fundamental $\underline{\mathbf{4}}$ of Sp_2 subject to a Majorana condition, as required by $N = (0,2)$ supersymmetry.

In this paper defining spinors globally on the five-brane is investigated. We shall consider the brane world-volume as an $N = 2$ chiral tensor theory, without reference to how or whether it is embedded in an background space.

If the five-brane should be a spin-manifold $w_2(TM) = 0$, then the spinor bundles of the normal and tangent bundles, $S(NM)$ and $S(TM)$, exist, and world-volume Fermions can be defined globally as sections of the tensor product bundle $S(TM) \otimes S(NM)$. If this is not the case and $w_2(TM) \neq 0$, a straightforward spin$^{\mathbb{C}}$ structure is of no use, as world-volume supersymmetry does not allow the Fermions to be electrically charged with respect to a U(1) gauge field. Indeed, there is no Abelian one-form in the massless spectrum of the theory. Instead of this, the Fermions are coupled to the spin group of the normal bundle Sp_2, and one is faced with a somewhat more general phenomenon.

JHEP09(2006)028

Given the embedding $\iota : M \hookrightarrow X$, it should be possible to lift the normal and the tangent bundles $NM \oplus TM$ to a spinor bundle together $S(NM \oplus TM)$, even though this bundle might not factorise in the form $S(TM) \otimes S(NM)$. The reason for this is the fact that the 11-dimensional bundle does exist, and one can always consider its pull-back bundle $\iota^* S(TX)$. Note, however, that the present paper does not make use of these arguments but, for reasons outlined in section 1, relies entirely on the intrinsic world-volume structure of the brane.

One way to see the effects of this is by noticing that on the M5-brane the 11-dimensional spin group $\mathrm{Spin}(1,10)$ is broken to $\mathrm{Sp}_2 \ltimes \mathrm{Spin}(1,5)$. This is a quotient of the direct product group by the equivalence $(g \cdot \tilde{a},\ \alpha) \sim (g,\ a \cdot \alpha)$ where $\tilde{a}$ and a generate a $\mathbb{Z}_2$ subgroup of the centre of each factor. This group is an example of generalised spin groups $\mathrm{Spin}_G(1,5)$ that fit in the exact sequence

$$1 \longrightarrow G \longrightarrow \mathrm{Spin}_G(1,5) \longrightarrow \mathrm{SO}(1,5) \longrightarrow 1\,. \tag{2.6}$$

It is useful to think of $\mathrm{Spin}_G(1,5)$ abstractly as an extension of $\mathrm{SO}(1,5)$ by G, in the same way as the standard spin group is an extension of the special orthogonal group by $\mathbb{Z}_2$. This paper concentrates on the specific extension that arises on an M5-brane: in the following

$$\mathrm{Spin}_G(1,5) \,=\, \mathrm{Sp}_2 \ltimes \mathrm{Spin}(1,5)\,, \tag{2.7}$$

where $G = \mathrm{Sp}_2$, is assumed throughout. Apart from this form of the generalised spin structure, no other information of the bulk theory is used in the calculation.

3. Characteristic classes

Recall that principal bundles $P \longrightarrow M$ with structure group H on a manifold M are in a one-to-one correspondence with the homotopy classes of mappings f from the manifold M to the corresponding classifying space BH. These mappings form the mapping class group $[M, BH]$. The bundle P is the pull back f^*EH of the universal H-bundle $EH \longrightarrow BH$. Generators of the cohomology of the classifying space $\omega \in H^*(BH)$ pull back to characteristic classes $f^*\omega$ of the bundle $P = f^*EH$. It is therefore of interest to determine the cohomology of $B\,\mathrm{Spin}_G(1,5)$.

The group $\mathrm{SO}(5)$ has one connected component, and $\mathrm{Spin}(5) = \mathrm{Sp}_2$ is its simply connected compact double cover. The modulo-two and integral cohomologies of the orthogonal group are

$$H^*(B\,\mathrm{SO}(5), \mathbb{Z}_2) \,=\, \mathbb{Z}_2[\bar{w}_2, \bar{w}_3, \bar{w}_4, \ldots] \tag{3.1}$$

$$H^*(B\,\mathrm{SO}(5), \mathbb{Z}) \,=\, \mathbb{Z}[\bar{p}_1, \bar{p}_2, \bar{W}_3, \bar{W}_5, \bar{W}_7']/\sim\,, \tag{3.2}$$

where the following equivalence should hold in the integral cohomology

$$(\bar{W}_7')^2 \,\sim\, (\bar{W}_3)^2\bar{p}_2 + (\bar{W}_5)^2\bar{p}_1\,. \tag{3.3}$$

Though the integral cohomology of spin groups is in general complicated [10], the isomorphism $\mathrm{Spin}(5) = \mathrm{Sp}_2$ implies the structure

$$H^*(B\,\mathrm{Sp}_2, \mathbb{Z}_2) = \mathbb{Z}_2[\bar{w}_4, \bar{w}_8] \tag{3.4}$$
$$H^*(B\,\mathrm{Sp}_2, \mathbb{Z}) = \mathbb{Z}[\bar{\lambda}_1, \bar{\lambda}_2] \tag{3.5}$$

as freely generated polynomial algebrae. The generators $\bar{w}_{4i}$ are modulo-two reductions of the integral generators $\bar{\lambda}_i$ [11, 12].

Given the fibration [13]

$$B\mathbb{Z}_2 \hookrightarrow B\,\mathrm{Sp}_2 \xrightarrow{\pi} B\,\mathrm{SO}(5)\,, \tag{3.6}$$

the following relations hold between generators of the cohomologies of $B\,\mathrm{SO}(5)$ and $B\,\mathrm{Sp}_2$: first of all, the notation is well-defined in the sense that the generators $\bar{w}_i$ of the latter are really pull-backs $\pi^*\bar{w}_i$ from the former; secondly, [12]

$$\pi^*\bar{p}_1 = 2\bar{\lambda}_1 \tag{3.7}$$
$$\pi^*\bar{p}_2 = 2\bar{\lambda}_2 + \bar{\lambda}_1^2\,. \tag{3.8}$$

The torsion classes in the integral cohomology are related to modulo-two generators by

$$\bar{W}_i = \beta(\bar{w}_i) \tag{3.9}$$
$$\bar{W}_7' = \beta(\bar{w}_2\bar{w}_4)\,. \tag{3.10}$$

The mapping β is the Bockstein of the modulo-two short exact sequence

$$1 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \longrightarrow 1\,. \tag{3.11}$$

If one denotes by r the reduction of integral classes modulo two, the following results [14] hold:

$$r(\bar{\lambda}_i) = \bar{w}_{2i}^2 \tag{3.12}$$
$$r \circ \beta = \mathrm{Sq}^1\,. \tag{3.13}$$

The group $\mathrm{SO}(1,5)$ has two connected components. The one connected to unity fits in the fibration $\mathbb{R}^5 \hookrightarrow \mathrm{SO}^0(1,5) \longrightarrow \mathrm{SO}(5)$. It is therefore contractible to $\mathrm{SO}(5)$. The maximal compact subgroup of $\mathrm{SO}(1,5)$ is $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(5)) = \mathbb{Z}_2 \times \mathrm{SO}(5)$. A group shares its classifying space with its maximal compact subgroup [11], so that

$$B\,\mathrm{SO}(1,5) = B(\mathbb{Z}_2 \times \mathrm{SO}(5)) = B\mathbb{Z}_2 \times B\,\mathrm{SO}(5)\,. \tag{3.14}$$

Given the Stiefel-Whitney classes $H^*(B\,\mathrm{SO}, \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, w_4, \ldots]$ and the results in integral cohomology of ref. [14], one obtains

$$H^*(B\,\mathrm{SO}(1,5), \mathbb{Z}_2) = \mathbb{Z}_2[\omega] \otimes \mathbb{Z}_2[w_2, w_3, w_4, w_5] \tag{3.15}$$
$$H^*(B\,\mathrm{SO}(1,5), \mathbb{Z}) = \mathbb{Z}[\varpi] \otimes \mathbb{Z}[p_1, p_2, W_3, W_5, W_7']/\sim\,. \tag{3.16}$$

The algebra generated in (3.16) is not free, but there is an equivalence similar to (3.3). The generator $\omega \in H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ is of degree one and $\varpi \in H^*(B\mathbb{Z}_2, \mathbb{Z})$ of degree two such that $2\varpi = 0$. Note that on a non-compact Lorentzian manifold, where more specifically the time direction is non-compact, the characteristic classes corresponding to pull-backs of these two generators are trivial.

The connected part of the world-volume spin group is $\mathrm{Spin}^0(1,5) = \mathrm{SL}(2, \mathbb{H})$. The full group has two components: one is connected to $\mathbf{1}$, the other is connected to the chirality operator χ. The maximal compact subgroup is Sp_2, so

$$B\,\mathrm{Spin}(1,5) \;=\; B\mathbb{Z}_2 \times B\,\mathrm{Sp}_2\;. \tag{3.17}$$

This implies

$$H^*(B\,\mathrm{Spin}(1,5), \mathbb{Z}_2) \;=\; \mathbb{Z}_2[\omega] \otimes \mathbb{Z}_2[w_4, w_8] \tag{3.18}$$
$$H^*(B\,\mathrm{Spin}(1,5), \mathbb{Z}) \;=\; \mathbb{Z}[\varpi] \otimes \mathbb{Z}[\lambda_1, \lambda_2]\,, \tag{3.19}$$

with $w_{4i} = r(\lambda_i)$.

Spinors on an M-theory five-brane carry both a representation of Sp_2 and $\mathrm{Spin}(1,5)$. This means that the *physical* extension $\mathrm{Spin}_G(1,5)$ is of the form

$$\mathbb{Z}_2 \hookrightarrow \mathrm{Sp}_2 \times \mathrm{Spin}(1,5) \longrightarrow \mathrm{Spin}_G(1,5)\,, \tag{3.20}$$

where the image of nontrivial element $-\mathbf{1}$ in the fibre is $(-\mathbf{1}, -\mathbf{1})$ in the total space. This leads to the fibration

$$B\mathbb{Z}_2 \hookrightarrow B\,\mathrm{Sp}_2 \times B\,\mathrm{Spin}(1,5) \longrightarrow B\,\mathrm{Spin}_G(1,5)\,. \tag{3.21}$$

Note that the $\mathbb{Z}_2$ factor that appears in (3.17) has a nontrivial image on the base. This means that the fibration (3.21) is nontrivial. Using the standard Leray-Serre spectral sequence, one finds the cohomology of the base space to be

$$\begin{aligned}&H^*(B\,\mathrm{Spin}_G(1,5), \mathbb{Z}_2)\\ &\quad= \mathbb{Z}_2[\omega] \otimes \mathbb{Z}_2[w_4, \bar{w}_4, w_8, \bar{w}_8] \otimes \mathbb{Z}_2[\omega_2, \omega_3, \omega_5, \omega_9, \ldots]\end{aligned} \tag{3.22}$$
$$\begin{aligned}&H^*(B\,\mathrm{Spin}_G(1,5), \mathbb{Z})\\ &\quad= \mathbb{Z}[\varpi] \otimes \mathbb{Z}[\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2] \otimes \mathbb{Z}[\varpi_3, \varpi_5, \varpi_9, \varpi_{17}, \ldots]\,,\end{aligned} \tag{3.23}$$

where in the last factor only generators ω_i (resp. ϖ_i) with degree of the form $i = 2^{r+1}$ appear.

This structure arises because (3.21) is a nontrivial fibration, and as such the transgression $d_2 : H^1(B\mathbb{Z}_2, \mathbb{Z}_2) \longrightarrow H^2(B\,\mathrm{Spin}_G(1,5), \mathbb{Z}_2)$ is nontrivial. On the level of the Leray-Serre spectral sequence this means first of all that the derivative d_2 acting on $E_2^{*,*}$ has to be nontrivial $d_2(a) = \omega_2 \neq 0$ where $a \in H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$ is the generator. By Corollary 6.9 of [15] this leads to a tower of generators

$$d_{2i+2}(a^{2^{i+1}}) = d_{2i+2}(\mathrm{Sq}^i\, a^{2^i}) = \mathrm{Sq}^i\, d_{i+1} a^{2^i} = \mathrm{Sq}^i\, \omega_{2^i+1} = \omega_{2^{i+1}+1}\,. \tag{3.24}$$

Though Corollary 6.9 of [15] does not apply over integers, a similar argument can be devised, and the above result follows.

The spectral sequence keeps track of the various cohomology groups not as rings but simply as graded vector spaces. Therefore, the generators ω_i and ϖ_i have, a priori, nothing to do with Stiefel-Whitney classes. The generators that are a priori related to Stiefel-Whitney classes appear in (3.18) and (3.19). The analysis of section 4 is required to clarify this issue, and it turns out that the new generators ω_i should a posteriori be identified with universal Stiefel-Whitney classes, at least up to degree five.

The fact that the cohomology involves an infinite tower of generators should not be surprising. First of all, Spin groups do have higher degree generators than SO groups; an example is the degree eight class of $\mathrm{Spin}(5) = \mathrm{Sp}_2$ whereas the generators of characteristic classes of SO(5) go only up to degree five. It turns in fact out ([16], Prop. 15.2) that $H^*(B\,\mathrm{Spin}(n), \mathbb{Z}_2)$ is a polynomial algebra precisely for $n \leq 9$. As $\mathrm{Spin}_G(1,5)$ is in a certain sense a reduced form of $\mathrm{Spin}(1,10)$, one should perhaps expect to find such a rich structure.

Section 5 provides a partial consistency check to these results.

4. Lifting

The exact sequence (2.6) induces the fibration [13] of classifying spaces

$$BG \hookrightarrow B\,\mathrm{Spin}_G(1,5) \longrightarrow B\,\mathrm{SO}(1,5)\,. \tag{4.1}$$

In what follows, the mapping class group $[M, B\,\mathrm{Spin}_G(1,5)]$ is analysed in terms of the Moore-Postnikov system of this fibration. These techniques were first introduced in the Physics Literature in [6].

The Moore-Postnikov system of the fibration (4.1) of the classifying space $E := B \times \mathrm{Spin}_G(1,5)$ consists of a sequence of fibrations

$$p_n : E^{[n]} \longrightarrow E^{[n-1]} \tag{4.2}$$

with fibre $K(\pi_n(BG), n) = K(\pi_{n-1}, n)$. At level $n = 0$ the space is simply the base space $E^{[1]} = B\,\mathrm{SO}(1,5)$; for each $n > 1$, $E^{[n]}$ has the homotopy groups $\pi_i(E)$ for $i \leq n$, and 0 otherwise. The larger the index n, the better approximation $E^{[n]}$ is of the total space E. Each fibration can be chosen in terms of the Postnikov invariants

$$\left[k^{n+1} : E^{[n]} \longrightarrow \hat{K}(\pi_{n-1}, n+1)\right], \tag{4.3}$$

which are homotopy classes of mappings. Using the isomorphism

$$[M, K(\pi, n)] \simeq H^n(M, \pi)\,, \tag{4.4}$$

these invariants can be considered elements of $H^{n+1}(E^{[n]}, \pi_{n-1})$.

In a non-simply connected case [17] the Eilenberg-MacLane spaces $\hat{K}(\pi_i, n)$ appearing in the Postnikov invariants are certain twisted versions of the standard spaces $K(\pi_i, n)$. In $H^n(M, \tilde{\pi}_i)$ the coefficient sheaf $\tilde{\pi}_i$ is the constant sheaf π_i twisted by the action of an element of $\mathrm{Aut}\,\pi_i$ over noncontractible paths in the total classifying space E.

In the present case there is precisely one such non-contractible path, as the total classifying space has the fundamental group $\pi_1(E) = \pi_0(\mathrm{Spin}_G(1,5)) = \mathbb{Z}_2$. Along these paths a tangent vector picks up a holonomy from the component of $\mathrm{SO}(1,5)$ connected to a total reflection $-\mathbf{1} \in SO(1,5)$. On the level of the spin group, this rotation lifts to $\pm\chi \in \mathrm{Spin}(1,5)$, the chirality operator.

On a Lorentzian manifold M the tangent bundle splits to an $\mathrm{O}(1) \times \mathrm{O}(n)$ bundle $TM \simeq V^- \oplus V^+$. In the orientable case, the obstruction $w_1(TM) = w_1^+ + w_1^- = 0$ relates the two logically distinct obstructions $w_1^\pm$ for either one of the vector bundles $V^\pm$ to be separately orientable. We shall refer to w_1^- as the obstruction to a *temporal orientation*. It has a chance to be nontrivial if the six-manifold M is compact.

As $B\,\mathrm{SO}(1,5)$ has the same homotopy type as E up to four-sceletons, one can replace E here by $B\,\mathrm{SO}(1,5)$, and the question of twisting coefficient sheaves $\tilde{\mathbb{Z}} = \mathbb{Z}_\omega$ reduces to a choice of an element in $\omega \in H^1(B\,\mathrm{SO}(1,5), \mathbb{Z}_2) = \mathbb{Z}_2$. The nontrivial element there is the obstruction to a temporal orientation $\omega = w_1^-$. The twisting, if any, is then done by using the temporal orientation sheaf as the coefficient sheaf in the cohomologies where the obstructions have their values.

The relevant part of the Moore-Postnikov system can be conveniently represented as the diagram

$$\begin{tikzcd}
K(\pi_4,5) \arrow[r] & E^{[6]} \arrow[d, "p_6"'] & \\
K(\pi_3,4) \arrow[r] & E^{[5]} \arrow[r, "k_6"] \arrow[d, "p_5"'] & \hat{K}(\pi_4,6) \\
 & B\,\mathrm{SO}(t,s) \arrow[r, "k_5"] & \hat{K}(\pi_3,5)\,.
\end{tikzcd} \tag{4.5}$$

We have denoted here $\pi_i = \pi_i(\mathrm{Sp}_2)$. Here the following facts have been used:

- For bundles on a six-manifold M it is sufficient to consider the tower up to six-sceletons $[M,E] = [M,E^{[6]}]$;
- Similarly, $[M, K(\pi,n)] = H^n(M,\pi) = 0$ for $n > 6$.
- The homotopy groups $\pi_0 = \pi_1 = \pi_2 = 0$ are trivial in all of the present constructions.

Taking the corresponding mapping class groups and putting in place $\pi_3 = \mathbb{Z}$ and $\pi_4 = \mathbb{Z}_2$, one finds the two exact sequences

$$\begin{aligned}
H^5(M,\mathbb{Z}_2) \longrightarrow \Big[M, B\,\mathrm{Spin}_G(1,5)\Big] \xrightarrow{p_{6*}} \Big[M, E^{[5]}\Big] \xrightarrow{k_{6*}} H^6(M,\mathbb{Z}_2) \\
H^4(M,\mathbb{Z}) \longrightarrow \Big[M, E^{[5]}\Big] \xrightarrow{p_{5*}} \Big[M, B\,\mathrm{SO}(1,5)\Big] \xrightarrow{k_{5*}} H^5(M,\tilde{\mathbb{Z}})\,.
\end{aligned}$$

There are no nontrivial automorphisms of $\mathbb{Z}_2$, so the twisting is trivial in $H^6(M,\mathbb{Z}_2)$.

5. Obstructions

In the two exact sequences of the last section the cohomology groups on the right are obstructions for lifting the tangent bundle TM, as represented by a class $[\zeta] \in [M, B\,\mathrm{SO}(1,5)]$, to an element of $[M, E^{[5]}]$ and then from there to a generalised spin bundle, as represented by a class in $[M, B\,\mathrm{Spin}_G(1,5)]$. The cohomology groups on the left describe the latitude in the lifting procedure, and can be thought of as classifying generalised spin structures.

The map k_5 determines a class in $[k_5] \in H^5(B\,\mathrm{SO}(1,5), \mathbb{Z})$. Given the mapping $\zeta : M \longrightarrow B\,\mathrm{SO}(1,5)$ corresponding to the tangent bundle $[\zeta] = TM$, one has

$$k_{5*}[\zeta] = [k_5 \circ \zeta] = \zeta^*[k_5]\,. \tag{5.1}$$

If this obstruction vanishes, there is a lift of ζ to $[\hat{\zeta}] \in [M, E^{[5]}]$ that satisfies $p_{5*}[\hat{\zeta}] = [\zeta]$. Similarly, to lift $\hat{\zeta}$ further to a class $\xi \in [M, E^{[6]}] = [M, B\,\mathrm{Spin}_G(1,5)]$, the obstruction

$$k_{6*}[\xi] = \xi^*[k_6] \tag{5.2}$$

must vanish.

In section 3 it has been shown that all the generators of the cohomology of the classifying space $B\,\mathrm{SO}(1,5)$, where one lifts a class $\zeta \in [M, B\,\mathrm{SO}(1,5)]$ from, are present in the cohomology of the classifying space $B\,\mathrm{Spin}_G(1,5)$, to whose mapping class group one is lifting it. As there are no such missing generators, there should not be universal obstructions, and it should always be possible to choose a lift of $\zeta = TM$ to a $\xi \in [M, B\,\mathrm{Spin}_G(1,5)]$ in such a way that $p_{5*} \circ p_{6*}[\xi] = [\zeta]$. The results of section 3 leave two questions open, however:

- It needs to be shown that the degree five generator ω_5 is indeed the Stiefel-Whitney class w_5 and not a new generator;
- The lifting procedure proceeds in two steps: it needs to be shown that there are no missing generators in the cohomology of $E^{[5]}$ that would then somehow be restituted in the cohomology of $E^{[6]}$.

In this section it is shown that the identification $\omega_5 = w_5$ is valid. There will indeed turn out to be space for an intermediate obstruction to lifting from $E^{[5]}$ to $E^{[6]}$, but it is shown below that this obstruction vanishes in the present case.

In the process of performing such a lift, one must make independent choices that amount to picking an element first from $H^4(M, \mathbb{Z})$ and then from $H^5(M, \mathbb{Z}_2)$. In order to investigate this process further, and to have a partial check for the results presented in section 3, the lifting procedure and the vanishing of the obstructions in the Moore-Postnikov system of section 4 is verified in detail in this section.

5.1 Integral obstructions

By Hurewicz Isomorphism, the first nontrivial integral cohomology group of $K(\mathbb{Z}, 4)$ is generated by the fundamental class I_4

$$H^4(K(\mathbb{Z},4), \mathbb{Z}) = \mathrm{Hom}(H_4(K(\mathbb{Z},4), \mathbb{Z}), \mathbb{Z}) \tag{5.3}$$

$$= \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}\,. \tag{5.4}$$

It follows from the Leray-Serre spectral sequence that $E^{[5]}$ has the same cohomology groups as $B\,\mathrm{SO}(1,5)$ in degrees one, two, and three. In particular, one should identify

$$W_3 = \varpi_3\,. \tag{5.5}$$

In degree four, the total space E is known to have one more generator $\bar{\lambda}_1$ than the base space $B\,\mathrm{SO}(1,5)$. As the only available extra generator is the fundamental class I_4 of the fibre, one must identify $\bar{\lambda}_1 = nI_4$ for some integer n. Other multiples kI_4 for $k < n$ would then have to be eliminated by setting $d_5(kI_4)$ to be nontrivial. This could be done consistently only for $n = 1, 2$. In next section one discovers, however, that $\iota_4 = r(I_4)$ remains in the cohomology, and at the very latest in studying modulo-two cohomology one discovers $n = 1$,

$$\bar{\lambda}_1 = I_4\,, \tag{5.6}$$

and $d_5(I_4) = 0$.

As none of the higher approximations $E^{[n]}$, $n > 5$ will change integral cohomology in degrees less than five, one has for $i \leq 4$

$$H^i(E, \mathbb{Z}_2) = F^i\Big(H^*(B\,\mathrm{SO}(1,5), \mathbb{Z}) \otimes \mathbb{Z}[\bar{\lambda}_1]\Big) \tag{5.7}$$

$$= F^i\Big(\mathbb{Z}[\varpi, W_3, \lambda_1, \bar{\lambda}_1]\Big)\,, \tag{5.8}$$

where F^i filters out the degree i part from the ring. This is fully consistent with (3.23).

Without knowing the fifth cohomology group of the fibre $H^5(K(\mathbb{Z}, 4), \mathbb{Z})$ one cannot write down the precise fifth cohomology group $H^5(E^{[5]}, \mathbb{Z})$. However, as it is known that the differential $d_5 = 0$ is trivial, it is clear that $H^5(E^{[5]}, \mathbb{Z})$ includes all generators of the base $H^5(B\,\mathrm{SO}(1,5), \mathbb{Z})$. The next differential d_6 could eliminate generators from the cohomology of the base, but only at degree six $H^6(B\,\mathrm{SO}(1,5), \mathbb{Z})$.

As $H^5(E^{[5]}, \mathbb{Z})$ contains all the classes of $H^5(B\,\mathrm{SO}(1,5), \mathbb{Z})$, there is no obstruction to the lift. Choosing such a lift requires making a choice, which amounts to picking the class

$$\hat{\zeta}^*\bar{\lambda}_1 \in H^4(M, \mathbb{Z}) \tag{5.9}$$

that corresponds to half the first Pontryagin class of the normal bundle.

5.2 Modulo-two obstructions

Up to degree six, the homotopy type of $E = B\,\mathrm{Spin}_G(1,5)$ is $E^{[6]}$. The approximations $E^{[5]}$ and $E^{[6]}$ fit in the fibrations

$$K(\mathbb{Z}_2, 5) \hookrightarrow E^{[6]} \longrightarrow E^{[5]} \tag{5.10}$$

$$K(\mathbb{Z}, 4) \hookrightarrow E^{[5]} \longrightarrow B\,\mathrm{SO}(1,5)\,. \tag{5.11}$$

The modulo-two cohomology groups of the fibres are

$$H^*(K(\mathbb{Z}_2, 5), \mathbb{Z}_2) = \mathbb{Z}_2[\mathrm{Sq}^I] = \mathbb{Z}_2[\iota_5, \mathrm{Sq}^1\,\iota_5, \mathrm{Sq}^2\,\iota_5, \ldots] \tag{5.12}$$

$$H^*(K(\mathbb{Z}, 4), \mathbb{Z}_2) = \mathbb{Z}_2[\mathrm{Sq}^J] = \mathbb{Z}_2[\iota_4, \mathrm{Sq}^2\,\iota_4, \mathrm{Sq}^3\,\iota_4, \ldots]\,, \tag{5.13}$$

where ι_n are the generators, and the multi-indices I and J are appropriately restricted [15].

As the lowest element ι_4 in $H^*(K(\mathbb{Z},4),\mathbb{Z}_2)$ is of degree four, it follows from the Leray-Serre spectral sequence that E has the same cohomology groups as $B\,\mathrm{SO}(1,5)$ in degrees one, two, and three. In particular, one should identify

$$\omega_2 = w_2 \tag{5.14}$$
$$\omega_3 = w_3\,. \tag{5.15}$$

In degree four, the total space E is known to have one more generator $\bar{w}_4$ than the base space $B\,\mathrm{SO}(1,5)$. The only available new generator in that degree is the fundamental class of the fibre ι_4. To keep it in the cohomology, it must be transitive in (5.11), that is $d_r(i_4) = 0$ for $r \geq 2$. We identify

$$\iota_4 = \bar{w}_4\,. \tag{5.16}$$

This is consistent with the fact that the Sp_2 class is a reduction of an integral class $\bar{\lambda}_1$ as so is $\iota_4 = r(I_4)$. Indeed, there is no $\mathrm{Sq}^1\,\iota_4$ generator in (5.13).

There is no new generator in degree five in the fibre; the next new generators are $\mathrm{Sq}^2\,\iota_4$ in degree six, and $\mathrm{Sq}^3\,\iota_4$ in degree seven. By choosing $d_7(\mathrm{Sq}^2\,\iota_4)$ suitably, one could either keep or eliminate this generator. As there is no generator of degree seven in the cohomology of the base space, however, one expects this differential to be trivial, and $\mathrm{Sq}^2\,\iota_4$ to remain in the cohomology; the same applies to the generator $\mathrm{Sq}^3\,\iota_4$ in degree seven. We shall see below that this is indeed the only consistent choice in these degrees. Filtering out the degrees already analysed $i \leq 7$, the modulo-two cohomology of $E^{[5]}$ is therefore

$$H^i(E^{[5]},\mathbb{Z}_2) = F^i\Big(H^*(K(\mathbb{Z},4),\mathbb{Z}_2)\otimes H^*(B\,\mathrm{SO}(1,5),\mathbb{Z}_2)\Big) \tag{5.17}$$
$$= F^i\Big(\mathbb{Z}_2[\bar{w}_4,\mathrm{Sq}^2\,\bar{w}_4,\mathrm{Sq}^3\,\bar{w}_4]\otimes\mathbb{Z}_2[\omega,w_2,w_3,w_4,w_5]\Big)\,. \tag{5.18}$$

The Leray-Serre spectral sequence of fibration (5.10) implies now that the cohomology of the total space $E^{[6]}$ and that of $E^{[5]}$ coincide in degrees up to and including four. At degree five there is, potentially, a new generator ι_5, which is the fundamental class of the fibre $K(\mathbb{Z}_2,5)$. As known from section 3, there should not be one, so that $d_6(\iota_5) \neq 0$ must be a nontrivial element in $H^6(E^{[5]},\mathbb{Z}_2)$. The free ring structure determines

$$d_6(\iota_5) = \mathrm{Sq}^2\,\bar{w}_4\,. \tag{5.19}$$

In hindsight, leaving $\mathrm{Sq}^2\,\bar{w}_4$ in the cohomology of $E^{[5]}$ was, therefore, justified. The choice of d_6 removes now ι_5 from degree five and both $\mathrm{Sq}^2\,\bar{w}_4$ and $\omega\iota_5$ from degree six.

There is a generator of degree six in the fibre $\mathrm{Sq}^1\,\iota_5$, and $d_6(\mathrm{Sq}^1\,\iota_5) = 0$. The nontrivial differential (5.19) implies also

$$d_7(\mathrm{Sq}^1\,\iota_5) = \mathrm{Sq}^1\,\mathrm{Sq}^2\,\bar{w}_4 \tag{5.20}$$
$$= \mathrm{Sq}^3\,\bar{w}_4\,. \tag{5.21}$$

This generator is indeed in the cohomology of $E^{[5]}$, but is now eliminated together with $\mathrm{Sq}^1\,\iota_5$ from the cohomology of $E^{[6]}$. We see, then, in hindsight that the generators $\mathrm{Sq}^2\,\bar{w}_4$,

JHEP09(2006)028

$\mathrm{Sq}^3\,\bar{w}_4$ were indeed both required in the cohomology of $E^{[5]}$ so that the generators $\iota_5, \mathrm{Sq}^1\,\iota_5$ could be eliminated from the cohomology of $E^{[6]}$ consistently with the requirements of section 3.

The rest of the generators $\mathrm{Sq}^I\,\iota_5$ of the cohomology of the fibre $K(\mathbb{Z}_2,5)$ have potentially nontrivial transgressions as well

$$d_{p+6}(\mathrm{Sq}^I\,\iota_5) \,=\, \mathrm{Sq}^I\,\mathrm{Sq}^2\,\iota_4\,. \tag{5.22}$$

Here p is the degree of the multi-index I. Due to the different structures of the two cohomologies (5.12) and (5.13), not all elements in the latter are in the image of d_{p+6}; this should account for the tower of generators found in (3.22), and amounts to a consistency check up to degree six for the results in section 3.

Therefore, filtering out degrees $i \le 6$ by F^i

$$\begin{aligned} H^i(E^{[6]},\mathbb{Z}_2) \,&=\, F^i\Big(H^i(E^{[5]},\mathbb{Z}_2)/_{\mathrm{Sq}^2\,\bar{w}_4=\mathrm{Sq}^3\,\bar{w}_4=0}\Big) && (5.23)\\ &=\, F^i\Big(\mathbb{Z}_2[\omega,w_2,w_3,w_4,\bar{w}_4,w_5]\Big)\,. && (5.24)\end{aligned}$$

The lowest generators inherited from the universal Stiefel-Whitney classes of $B\,\mathrm{SO}(1,5)$, namely w_2, w_3, w_4, had no chance of getting eliminated in the above procedure. The fact that there was an extra generator $\bar{w}_4$ in degree six meant that one could not eliminate the fifth Stiefel-Whitney class w_5 either. One identifies, then,

$$\omega_5 \,=\, w_5\,. \tag{5.25}$$

It is interesting to note that though $\mathrm{Sq}^1\,\bar{w}_4 = 0$ in the fibre in this construction, a nontrivial class $\mathrm{Sq}^1\,w_4 = w_5$ is allowed on the base $B\,\mathrm{SO}(1,5)$.

The only characteristic class in (5.24) that is not inherited from the cohomology of $B\,\mathrm{SO}(1,5)$ is $\bar{w}_4$, the modulo-two generator of the cohomology of the $K(\mathbb{Z},4)$-fibre. As seen in section 5.1 in particular, it is there already in $H^*(E^{[5]},\mathbb{Z}_2)$.

One is now in a position to comment on the lifting procedure from $[M,E^{[5]}]$ to $[M,E]$. There is precisely one generator in degree six that is present in $E^{[5]}$ but not in $E^{[6]}$ due to (5.19). This generator is the obstruction, and consistency requires

$$\hat{\zeta}^*\,\mathrm{Sq}^2\,\bar{w}_4 \,=\, 0\,. \tag{5.26}$$

Recall that Steenrod squares commute with pull-backs. We can use the π^* of (3.6) in a given fibre

$$\mathrm{Sq}^2\,\bar{w}_4 = \mathrm{Sq}^2\,\pi^*\bar{w}_4 = \pi^*\,\mathrm{Sq}^2\,\bar{w}_4 = \pi^*(\bar{w}_2\bar{w}_4+\bar{w}_6) = 0\,, \tag{5.27}$$

because both $\bar{w}_2$ and $\bar{w}_6$ pull back to zero in the cohomology of $B\,\mathrm{Sp}_2$.

There is then no obstruction to lifting an element of $[M,E^{[5]}]$ to $[M,E]$. Such a lift requires making a choice; the latitude in different choices corresponds to elements of $H^5(M,\mathbb{Z}_2)$. Fixing this latitude does not correspond to choosing a characteristic class of $B\,\mathrm{Spin}_G(1,5)$, as opposed to the one performed in (5.9).

6. Noncompact cases

Thus far it was assumed only that $M \subset X$ is a *smooth orientable Lorentzian six-manifold*. For simplicity, in this section only, it is assumed to be also connected; the existence and classification of global spinors can clearly be discussed component by component if it is not.

In this section, and in this section only, it is further assumed that the five-brane world-volume M is *contractible* to some compact manifold M_n of dimension n. This happens, for instance, when the total world-volume is of the form

$$\mathbb{R}^{6-n} \hookrightarrow M \longrightarrow M_n \, . \tag{6.1}$$

This simplifies matters, as the homotopy axiom of cohomology — which holds also with local, or twisted, coefficients — guarantees $H^*(M) = H^*(M_n)$. There are four basic cases:

- At $n = 6$ the world-volume, including the time direction, is compact. To analyse this case the full power of the results of the previous sections are needed.
- At $n = 5$ there is one non-compact direction that one can think of as time. This case is discussed briefly in this section without recourse to the detailed structure of $B\,\mathrm{Spin}_G(1,5)$.
- At $n = 4$ the obstructions vanish obviously, and the lifts are classified in $H^4(M,\mathbb{Z}) = \mathbb{Z}$.
- For $n \leq 3$, the obstructions vanish, and there is a canonical lift.

Consider a class $\zeta \in [M, B\,\mathrm{SO}(1,5)]$ in the case $n = 5$, so that M is contractible to a five-dimensional compact manifold W: then

$$H^6(M,\mathbb{Z}_2) = 0 \tag{6.2}$$

$$H^5(M,\mathbb{Z}_2) = \mathbb{Z}_2 \tag{6.3}$$

$$H^5(M,\tilde{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{trivial twist } \zeta^*\omega = 0 \\ \mathbb{Z}_2 & \text{non-trivial twist}\,, \end{cases} \tag{6.4}$$

where $\omega \in H^1(B\,\mathrm{Spin}_G, \mathbb{Z}_2)$, and the class $\zeta^*\omega \in H^1(M,\mathbb{Z}_2)$ determines the character of the time orientation sheaf.

If one takes the non-compact direction to be time, then $\zeta^*\omega = 0$, and the fifth cohomology is taken over trivially twisted coefficients.

The homomorphism k_{5*} is defined on the fifth cohomology of the classifying space $H^5(B\,\mathrm{SO}(1,5))$, whose elements are all two-torsion $2H^5(B\,\mathrm{SO}(1,5)) = 0$. The target space, however, has no torsion elements $H^5(M,\mathbb{Z}) = \mathbb{Z}$. It follows that the mapping must be trivial $k_{5*} = 0$. There is then no obstruction to lifting the tangent bundle to a generalised spin bundle when the brane is compact but the time-direction is not.

Note that if the time direction is compact and the cohomology is nontrivially twisted, one must make recourse to the arguments in the rest of the paper to show the vanishing of the obstruction.

7. Discussion

We have shown that on an oriented Lorentzian five-brane world-volume there is no obstruction for lifting the tangent bundle to a generalised spin bundle with structure group $\mathrm{Spin}_G(1,5) = \mathrm{Sp}_2 \ltimes \mathrm{Spin}(1,5)$. In the process of eliminating obstructions, the cohomology of the classifying space of generalised spin bundles was found

$$\begin{aligned} &H^*(B\,\mathrm{Spin}_G(1,5), \mathbb{Z}_2) \\ &\quad = \mathbb{Z}_2[\omega] \otimes \mathbb{Z}_2[w_4, \bar{w}_4, w_8, \bar{w}_8] \otimes \mathbb{Z}_2[w_2, w_3, w_5, \ldots] \qquad (7.1) \\ &H^*(B\,\mathrm{Spin}_G(1,5), \mathbb{Z}) \\ &\quad = \mathbb{Z}[\varpi] \otimes \mathbb{Z}[\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2] \otimes \mathbb{Z}[W_3, W_5, \ldots]\,. \qquad (7.2) \end{aligned}$$

As a part of the considerations of section 5, we identified the degree five generator in the former cohomology ring as a Stiefel-Whitney class. Though the structure of these cohomologies as freely generated rings follows from the analysis, the action of the Steenrod algebra, or the modulo-two-reduction properties of integral generators, does not necessarily follow from those valid for Sp_2 or $\mathrm{Spin}(1,5)$.

The appearance of the second and the third Stiefel-Whitney classes there means that the underlying orientable manifold does not need to be spin or even spin$^{\mathbb{C}}$. The corresponding classes for the normal bundle are absent, effectively because it was assumed that $\mathrm{Spin}_G(1,5)$ was the extension of $\mathrm{SO}(1,5)$ by the spin group of the normal bundle. Note that this is justified already by the *local* structure of the world-volume theory as a six-dimensional $N = 2$ chiral tensor theory.

In order to specify such a lift, one has to choose a generalised spin structure. This amounts, effectively, to picking certain classes in the cohomology of the brane. More precisely, generalised spin structures are classified in

$$H^4(M, \mathbb{Z}) \oplus H^5(M, \mathbb{Z}_2)\,, \qquad (7.3)$$

in the sense that any two lifts differ by structure that can be characterised fully by an element in (7.3). One may ask whether these classes could be interpreted in terms of fixing characteristic classes of an Sp_2 (or an $\mathrm{SO}(5)$) bundle with which the twisting is done. As the structure group of the generalised spin bundle concerns really the semi-direct product of this group and $\mathrm{Spin}(1,5)$, one cannot, in general, translate characteristic classes of the total $\mathrm{Spin}_G(1,5)$ bundle to characteristic classes of the normal bundle.

In the special case where the Sp_2 bundle does exist as a lift of a vector bundle, say NM, one can give such an identification, though. Indeed, given such a (trivially) generalised spin bundle $\xi \in [M, B\,\mathrm{Spin}_G(1,5)]$ and $\hat{\zeta} = p_{6*}\xi$, the degree four part of the generalised spin structure $\hat{\zeta}^*\bar{\lambda}_1$ would correspond to the half Pontryagin class $\lambda(NM) = p_1(NM)/2$ of the underlying vector bundle. This determines the degree four class $w_4(NM) = r(\lambda(NM))$. The remaining piece of generalised spin structure does not seem to be related directly to characteristic classes of the normal bundle, not even in such an hypothetical case as above.

7.1 Spin structure in the bulk

Thus far using information from the bulk has been carefully avoided. It is interesting to note, nevertheless, that when the 11-dimensional background is spin, one can define also there the half Pontryagin class $\lambda(TX)$ that was responsible for shifts in the quantisation condition in the bulk in [3]. This gives the integral characteristic class

$$\lambda(TM) \ := \ i^*\lambda(TX) - \hat{\zeta}^*\bar{\lambda}_1 \,, \tag{7.4}$$

even though M does not need to be spin. The terminology is justified as, if the Sp_2 and $\mathrm{Spin}(1,5)$ bundles did exist independently, this would be the pertinent half Pontryagin class.

The existence of such a class would seem to indicate that its reduction modulo two should yield the class $w_4(TM)$. This is certainly true when the five-brane is spin. If this is indeed the case, then the topology of such a five-brane embedding is characterised by the constraint [18]

$$W_5(TM) = 0 \,, \tag{7.5}$$

as this is equivalent to $w_4(TM)$ being a reduction of an integral class.

Equation (7.5) could have arisen as a standard obstruction in the discussion of section 5.1, but did not; it is rather a consequence of the geometry of the embedding $\iota : M \hookrightarrow X$, than the existence of global twisted spinors on the five-brane. The constraint (7.5) should therefore be compared rather to the fact [19] that the bulk geometry satisfies $W_7(TX) = 0$, than that the obstruction to $\mathrm{spin}^{\mathbb{C}}$ structure would happen to have been $W_3(TM)$.

7.2 Open problems

In the present case, where the generalised spin group is specifically fixed to be the physically relevant semidirect product $\mathrm{Sp}_2 \ltimes \mathrm{Spin}(1,5)$, the vanishing of all obstructions is automatic (other than the embedding-related (7.5)). For other extensions $\mathrm{Spin}_G(1,5)$ that fit in (2.6), the cohomology could be different. In particular, if the class W_5 should then be absent from the cohomology of $B\,\mathrm{Spin}_G(1,5)$, it would appear as the obstruction in (7.5).

There are many open problems with the dynamics and geometry of five-branes, as there are several approaches to describing them. One of them is in terms of embedded smooth submanifolds, others include for instance M(atrix) theory constructions [20]. The point of view taken in this paper extends the class of geometrically described brane configurations by showing that some structures, such as generalised spin structure, make sense on the brane quite irrespective of the details of how the brane is coupled to the bulk.

It would be interesting to relax further some of the assumptions made in the beginning, namely that spaces involved should be orientable. In fact, the bulk M-theory is known to possess a reflection symmetry [9, 21]; also the fact that the world-volume theory involves self-dual three-forms may mean that requiring a Lagrangean formulation in terms of a local action integral may be too restrictive. Among other matters, this would lead to

JHEP09(2006)028

more complicated twists in the cohomology where the obstructions take there values and, perhaps, provide insights in five-branes as quantum mechanical solitonic objects in M-theory.

Finally, it is interesting to note that in their recent work, where they construct partition functions for anti-self-dual tensor theories such as the five-brane world-volume theory, Belov and Moore [22] make use of structures that involve choices differing by elements $\mu \in H^4_{\mathrm{tors}}(M, \mathbb{Z})$. It would be interesting to understand precisely the relationship of that structure to the choice of a generalised spin structure in the present paper.

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References

[1] P.S. Howe, G. Sierra and P.K. Townsend, *Supersymmetry in six-dimensions*, *Nucl. Phys.* **B 221** (1983) 331.

[2] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string/string duality: a one-loop test*, *Nucl. Phys.* **B 452** (1995) 261 [`hep-th/9506126`].

[3] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [`hep-th/9609122`].

[4] P.S. Howe, E. Sezgin and P.C. West, *Covariant field equations of the M-theory five-brane*, *Phys. Lett.* **B 399** (1997) 49 [`hep-th/9702008`].

[5] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, *Phys. Lett.* **B 76** (1978) 409.

[6] S.J. Avis and C.J. Isham, *Generalized spin structures on four-dimensional space-times*, *Commun. Math. Phys.* **72** (1980) 103.

[7] D. Freed, J.A. Harvey, R. Minasian and G.W. Moore, *Gravitational anomaly cancellation for M-theory fivebranes*, *Adv. Theor. Math. Phys.* **2** (1998) 601 [`hep-th/9803205`].

[8] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [`hep-th/9711200`].

[9] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [`hep-th/9610234`].

[10] D. Quillen, *The* mod 2 *cohomology rings of extra-special* 2*-groups and the spinor groups*, *Math. Ann.* **194** (1971) 197.

[11] H.B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, 1989.

[12] E. Thomas, *On the cohomology group of the classifying space for the stable spinor group*, *Bol. Soc. Mat. Mexicana* **7** (1962) 57.

[13] A. Borel, *Topics in the homology theory of fibre bundles*, *Lect. Notes in Math.* **36** (1967) 1.

[14] E.H. Brown Jr., *Cohomology of $B\,\mathrm{SO}_n$ and $B\mathrm{O}_n$ with integer coefficients*, *Proc. Amer. Math. Soc.* **85** (1982) 283.

[15] J. McCleary, *A user's guide to spectral sequences*, Cambridge University Press 2001.

[16] A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, *Amer. J. Math.* **76** (1954) 273.

[17] C.A. Robinson, *Moore-Postnikov systems for non-simple fibrations*, *Illinois J. Math.* **16** (1972) 234.

[18] J. Kalkkinen, *Holonomies of intersecting branes*, *Fortschr. Phys.* **53** (2005) 913 [hep-th/0412166].

[19] D.-E. Diaconescu, G.W. Moore and E. Witten, *E_8 gauge theory and a derivation of K-theory from M-theory*, *Adv. Theor. Math. Phys.* **6** (2003) 1031 [hep-th/0005090].

[20] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[21] E. Witten, *Five-branes and M-theory on an orbifold*, *Nucl. Phys.* **B 463** (1996) 383 [hep-th/9512219].

[22] D. Belov and G.W. Moore, *Holographic action for the self-dual field*, hep-th/0605038.